Noncommutative Network Models

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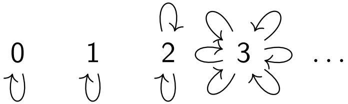
The Symmetric Groupoid



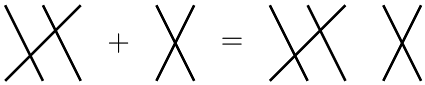
Definition

Let S *denote the category with finite sets* $\mathbf{n} = \{1, \ldots, n\}$ *as objects, and bijections for morphisms.*

The Symmetric Groupoid

FinSet is cocartesian monoidal, and restricting this structure to S is still symmetric monoidal, just not cocartesian

- $n + m$ is exactly what you think
- $\sigma + \tau$ looks like



Network Models

Definition

A **Network Model** *is a lax symmetric monoidal functor* $F\colon \mathsf{S} \to \mathsf{Cat}$*. Let* NetMod *denote the category with network models for objects, and monoidal natural transformations for morphisms.*

Simple Graphs

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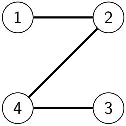
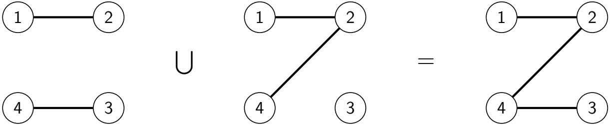


is the set $\{\{1,2\},\{2,4\},\{3,4\}\}$ in this setting.

Graph Monoids

Defining graphs this way allows us to take unions of graphs



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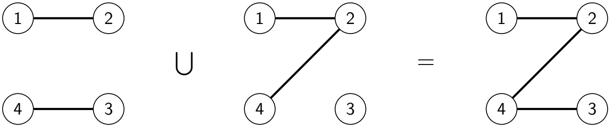

so the set of simple graphs on $\mathbf{n}$, denoted $\mathrm{SG}(\mathbf{n})$, is a monoid with operation $\cup$.

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Putting them all together gives a functor

$$\mathrm{SG} \colon \mathsf{S} \to \mathsf{Mon}$$

Given two graphs, g in $\mathrm{SG}(\mathbf{n})$ and h in $\mathrm{SG}(m)$, the disjoint union $g \sqcup h$ is a graph in $\mathrm{SG}(n+m)$.

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which makes SG a symmetric lax monoidal functor

$$(\mathrm{SG}, \sqcup) \colon (\mathsf{S}, +) \to (\mathsf{Mon}, \times)$$

Other Examples of Network Models

- Multigraphs
- Directed Graphs
- Partitions
- Graphs with colored vertices
- Petri Nets

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- Multigraphs
- Directed Graphs
- Partitions
- Graphs with colored vertices
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- Graphs with edges weighted by a monoid

Combinatorics Captured by the Definition

1. $e_n \cup g = g = g \cup e_n$
2. $g_1 \cup (g_2 \cup g_3)$
3. $\sigma(g_1 \cup g_2) = \sigma g_1 \cup \sigma g_2$
4. $\sigma e_n = e_n$
5. $(\sigma_1 \sigma_2) g = \sigma_1(\sigma_2 g)$
6. $1(g) = g$

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6. $1(g) = g$
7. $(g_1 \cup g_2) \sqcup (h_1 \cup h_2) = (g_1 \sqcup h_1) \cup (g_2 \sqcup h_2)$
8. $e_m \sqcup e_n = e_{m+n}$
9. $\sigma g \sqcup \tau h = (\sigma + \tau)(g \sqcup h)$
10. $g_1 \sqcup (g_2 \sqcup g_3) = (g_1 \sqcup g_2) \sqcup g_3$
11. $e_0 \sqcup g = g \sqcup e_0$
12. $B_{m,n}(h \sqcup g) = g \sqcup h$

Ordinary Network Models

$\mathrm{SG}(\mathbf{n})$ is the monoid whose underlying set is $2^{\binom{n}{2}}$ and whose monoid operation is union. More succinctly, $\mathrm{SG}(\mathbf{n}) = \mathbb{B}^{\binom{n}{2}}$, where $\mathbb{B}$ is the boolean monoid.

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Theorem

Given a monoid M, there is a network model Γ_M with $\Gamma_M(\mathbf{n}) = M^{\binom{n}{2}}$, and the actions of the symmetric groups given by "permuting the vertices". Moreover, this gives a functor $\Gamma\colon \mathsf{Mon} \to \mathsf{NetMod}$.

Examples of Γ_M

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- Let $M = \mathbb{B}_m$ be the monoid with elements $\{0, \dots, m\}$, and the operation defined by $j + k = \min\{j + k, m\}$. Then Γ_M is the network model for graphs with no more than m edges.

Range-Limited Networks

A **range-limited network** in a metric space (X, d) is a simple graph G with vertex set $\mathbf{n}$ and a function $f : \mathbf{n} \to X$.

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Let $L_k(\mathbf{n})$ denote the set of k-limited networks with vertex set $\mathbf{n}$.

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Networks of Bounded Degree

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Let $B_k(\mathbf{n})$ be the set of graphs with degree $\leq k$.

Eckmann-Hilton for Network Models

Let $(F, \sqcup)$ be a network model, $a \in F(n)$ and $b \in F(m)$. Then

$$\begin{aligned}(a \sqcup e_m) \cup (e_n \sqcup b) &= (a \cup e_n) \sqcup (e_m \cup b) \\ &= (e_n \cup a) \sqcup (b \cup e_m) \\ &= (e_n \sqcup b) \cup (a \sqcup e_m)\end{aligned}$$

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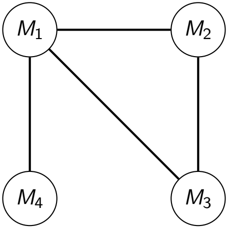
So there are relative commutativity relations that must hold in each monoid of a network model.

Graph products of monoids



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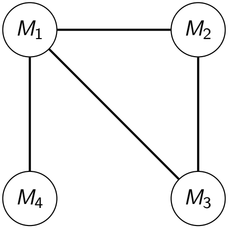

$$G(\{M_i\}) = \coprod M_i / R$$

Kneser Graphs

The Kneser graph $KG_{n,m}$ has vertex set $\binom{n}{m}$, the set of m-element subsets of an n-element set, and an edge between two vertices if they are disjoint subsets.

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The graphs $KG_{n,2}$ are perfect for describing the disjointness-commutativity that network models demand!

Free Network Models

Theorem

The functor NetMod $\to$ Mon *defined by* $F \mapsto F(2)$ *has a left adjoint* $\Gamma_{-,\mathrm{Mon}} \colon \mathrm{Mon} \to \mathrm{NetMod}$, *with* $\Gamma_{M,\mathrm{Mon}}(\mathbf{n}) = KG_{n,2}(M)$.

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This network model is able to remember the order in which edges were added to a network

Further reading and acknowledgments

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