Monoidal Grothendieck Construction

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My papers

- *Network Models* (2017)
  w/ John Baez, John Foley, and Blake Pollard
- *Noncommutative Network Models* (2018)
- *Monoidal Grothendieck Construction* (2019)
  w/ Christina Vasilakopoulou
- *Network Models from Petri Nets with Catalysts* (2019)
  w/ John Baez and John Foley

Outline

1. Motivation
2. 2-categorical preliminaries
3. The Grothendieck Construction
4. Monoidal Grothendieck Construction
5. Future Work

Motivation

$$k \in \mathsf{Ring}$$

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$$k \in \mathsf{Ring} \rightsquigarrow \mathsf{Mod}_k \in \mathsf{Cat}$$

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$$\mathsf{Mod}_{all}???$$

Motivation

Grothendieck: Yes!

- objects (k, M), where $M \in \mathrm{Mod}_k$
- maps $(f, g)\colon (k, M) \to (k', M')$ where $f\colon k \to k'$ and $g\colon M \to f^*(M')$

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$$k \in \mathsf{Ring} \rightsquigarrow \mathsf{Mod}_k \in \mathsf{Cat}$$

$$\mathsf{Mod}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$$

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Motivation

Given

$$\mathsf{Mod}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$$

we defined Mod_{all} to have

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Given

$$\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

we define $\int \mathcal{F}$ to have

- objects (x, a), where $x \in \mathcal{X}$, $a \in \mathcal{F}(x)$
- maps $(f, g)\colon (x, a) \to (x', a')$ where $f\colon x \to x'$ and $g\colon a \to \mathcal{F}f(a')$

2-categories

A **2-category** has:

- objects $\mathrm{ob}\,\mathcal{C}$
- a set $\mathcal{C}(x, y)$ of 1-morphisms between pair of objects $x, y \in \mathrm{ob}\,\mathcal{C}$
- a set $\mathcal{C}(f, g)$ of 2-morphisms between pair of 1-morphisms $f, g \in \mathcal{C}(x, y)$

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- composition of 1-morphisms
- horizontal and vertical composition of 2-morphisms

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satisfying:

- everything is associative and unital
- interchange law for 2-morphisms

2-categories

Alternatively, a **2-category** is a category enriched in categories:

- $\mathcal{C}(x, y)$ is a hom-category
- composition $\circ\colon \mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$ is a functor
- interchange law relates the (vertical) composition in $\mathcal{C}(x, y)$ to the (horizontal) composition functor $\circ$

Prototypical Example: Cat

The 2-category Cat has:

- categories for objects
- functors for 1-morphisms
- natural transformations for 2-morphisms

Example: Trivial 2-morphisms

Any 1-category $\mathcal{C}$ can be converted into a 2-category by considering each $\mathcal{C}(x, y)$ to be a discrete category.

2-functors

A **2-functor** between 2-categories $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is:

- a function $\mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$
- for every pair of objects $x, y \in \mathrm{ob}\,\mathcal{C}$, a functor $\mathcal{C}(x, y) \to \mathcal{D}(\mathcal{F}x, \mathcal{F}y)$

such that

- $\mathcal{F}(1_x) = 1_{\mathcal{F}(x)}$
- $\mathcal{F}(f \circ g) = \mathcal{F}f \circ \mathcal{F}g$
- horizontal composition: $\mathcal{F}(\alpha \circ \beta) = \mathcal{F}\alpha \circ \mathcal{F}\beta$

Pseudofunctors

A **pseudofunctor** between 2-categories $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$ is:

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with

- $\phi_0 \colon \mathcal{F}(1_x) \xrightarrow{\sim} 1_{\mathcal{F}(x)}$ 2-isomorphism
- $\phi \colon \mathcal{F}(f \circ g) \xrightarrow{\sim} \mathcal{F}f \circ \mathcal{F}g$ 2-isomorphism

such that

- horizontal composition: $\mathcal{F}(\alpha \circ \beta) = \mathcal{F}\alpha \circ \mathcal{F}\beta$

Indexed Categories

Let $\mathcal{X}$ be a category.

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- An **indexed natural transformation** is a modification $m \colon \alpha \Rrightarrow \beta$
- This defines a 2-category $\mathsf{ICat}(\mathcal{X})$.

Example: Rings and Modules

A ring homomorphism $f\colon k \to k'$ induces a functor

$$f^*\colon \mathrm{Mod}_{k'} \to \mathrm{Mod}_k$$

given by $f^*(M) = M$ but with the k-action defined by

$$r.m = f(r).m$$

for $r \in k$, and

$$f^*(g) = g$$

This gives a functor $\mathrm{Mod}\colon \mathrm{Ring}^{\mathrm{op}} \to \mathrm{Cat}$ sending a ring to its category of modules and a morphism f to $\mathrm{Mod}_f = f^*$.

Fibred Categories

$$P\colon \mathcal{A} \to \mathcal{X}$$

$\phi\colon a \to b$ in $\mathcal{A}$ is **cartesian** if given

- $g\colon x' \to x$ in $\mathcal{X}$
- $\theta\colon a' \to b$ in $\mathcal{A}$
- with $P\theta = f \circ g$

then $\exists!\,\psi\colon a' \to a$ such that $P\psi = g$ and $\theta = \phi \circ \psi$

$$\begin{tikzcd}
a' \arrow[dr, dashed, "\exists!\psi"'] \arrow[drr, "\theta"] \arrow[dd, dotted] & & & \\
& a \arrow[r, "\phi"'] \arrow[dd, dotted] & b \arrow[dd, dotted] & \text{in } \mathcal{A} \\
x' \arrow[dr, "g"'] \arrow[drr, "f\circ g = P\theta"] & & & \\
& x \arrow[r, "f = P\phi"'] & y & \text{in } \mathcal{X}
\end{tikzcd}$$

Fibred Categories

- For $x \in \mathrm{ob}\,\mathcal{X}$, the **fibre of *x*** is $\mathcal{A}_x := P^{-1}(x)$.
- $P\colon \mathcal{A} \to \mathcal{X}$ is called a **fibration** if and only if, for all $f\colon x \to y$ in $\mathcal{X}$ and $b \in \mathcal{A}_Y$, there is a **cartesian lifting** of b along f, $\phi\colon a \to b$.
- The category $\mathcal{X}$ is then called the **base** of the fibration, and $\mathcal{A}$ its **total category**.

2-Category of Fibrations

The 2-category $\mathrm{Fib}(\mathcal{X})$:

- an object is a fibration $P \colon \mathcal{A} \to \mathcal{X}$
- a 1-morphism is a functor F

$$\begin{tikzcd}
\mathcal{A} \arrow[dd, "F"'] \arrow[dr, "P"] & \\
& \mathcal{X} \\
\mathcal{B} \arrow[ur, "Q"'] &
\end{tikzcd}$$

  which preserves cartesian liftings

- a 2-morphism is a natural transformation

$$\begin{tikzcd}
\mathcal{A} \arrow[dd, bend right, "F"'] \arrow[dd, bend left, "F'"] \arrow[dd, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, "P"] & \\
& \mathcal{X} \\
\mathcal{B} \arrow[ur, "Q"'] &
\end{tikzcd}$$

Example: Graphs

Let Grph denote the category of directed multi-graphs. Define the vertex functor

$$\mathrm{V} \colon \mathsf{Grph} \to \mathsf{Set}$$

by sending a graph to its set of vertices, and a map of graphs to its vertex component.
V is a fibration.

The Grothendieck Construction

In SGA I, Grothendieck described a construction for a fibred category from an indexed category.

The Grothendieck Construction

Given an indexed category

$$\mathcal{F} \colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

define a category $\int \mathcal{F}$

- objects (x, a) where $x \in \mathcal{X}$ and $a \in \mathcal{F}x$

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- morphisms $(f, k)\colon (x, a) \to (y, b)$ where $f\colon x \to y$ in $\mathcal{X}$, and $k\colon a \to \mathcal{F}f(b)$ in $\mathcal{F}x$
- composition $(g, \ell) \circ (f, k)\colon (x, a) \to (y, b) \to (z, c)$ is given by $g \circ f\colon a \to b \to c$ in $\mathcal{X}$ and

$$a \xrightarrow{k} (\mathcal{F}f)(b) \xrightarrow{(\mathcal{F}g)(\ell)} (\mathcal{F}g \circ \mathcal{F}f)(c) \xrightarrow{(\delta_{f,g})_c} \mathcal{F}(g \circ f)(c) \quad \text{in } \mathcal{F}x;$$

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- unit $1_{(x,a)}\colon (x, a) \to (x, a)$ is given by $1_x\colon x \to x$ in $\mathcal{X}$ and

$$a = 1_{\mathcal{F}x}a \xrightarrow{(\gamma_x)_a} (\mathcal{F}1_x)(a) \quad \text{in } \mathcal{F}x.$$

The Grothendieck Construction

$\int \mathcal{F}$ is naturally fibred over $\mathcal{X}$:

$$\begin{aligned} P_{\mathcal{F}} \colon \int \mathcal{F} &\to \mathcal{X} \\ (x, a) &\mapsto x \\ (f, k) &\mapsto f \end{aligned}$$

2-Equivalence

Theorem

This construction gives a 2-equivalence.

$$\mathsf{ICat}(\mathcal{X}) \cong \mathsf{Fib}(\mathcal{X})$$

Example: semidirect product

$$A\colon G \to \mathrm{Aut}(H)$$

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$$* \mapsto H$$

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$$\int A = H \rtimes G$$

Monoidal 2-categories

A **monoidal 2-category** $\mathcal{C}$ is a 2-category equipped with a pseudofunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $I \colon \mathbf{1} \to \mathcal{C}$ which are associative and unital up to coherent equivalence.

Examples of monoidal 2-categories

Both $\mathsf{ICat}(\mathcal{X})$ and $\mathsf{Fib}(\mathcal{X})$ are monoidal 2-categories under their cartesian monoidal structure.

Products in $\mathsf{ICat}(\mathcal{X})$

$$\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$
$$\mathcal{G}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

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Products in ICat($\mathcal{X}$)

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$$\mathcal{G}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

$$\mathcal{F} \boxtimes \mathcal{G}(x) = \mathcal{F}(x) \times \mathcal{G}(x)$$

$$\mathcal{X}^{\mathrm{op}} \xrightarrow{\Delta} \mathcal{X}^{\mathrm{op}} \times \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathcal{F}\times\mathcal{G}} \mathsf{Cat} \times \mathsf{Cat} \xrightarrow{\times} \mathsf{Cat}$$

Products in $\mathrm{Fib}(\mathcal{X})$

$$
\begin{tikzcd}
\mathcal{A} \times_{\mathcal{X}} \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dr, dashed, bend left, "P \boxtimes Q"] & \mathcal{A} \arrow[d, "P"] \\
\mathcal{B} \arrow[r, "Q"'] & \mathcal{X}
\end{tikzcd}
$$

Pseudomonoids

A **pseudomonoid** in a monoidal 2-category $(\mathcal{C}, \otimes, I)$ is an object a equipped with multiplication $m \colon a \otimes a \to a$, unit $j \colon I \to a$, and invertible 2-cells

$$\begin{tikzcd}
a \otimes a \otimes a \arrow[r, "1 \otimes m"] \arrow[d, "m \otimes 1"'] \arrow[dr, phantom, "\overset{\alpha}{\cong}"] & a \otimes a \arrow[d, "m"] \\
a \otimes a \arrow[r, "m"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \otimes I \arrow[r, "1 \otimes j"] \arrow[dr, "\sim"'] & a \otimes a \arrow[d, "m"] \arrow[dl, phantom, "\overset{\lambda}{\cong}"] \arrow[dr, phantom, "\overset{\rho}{\cong}"] & I \otimes a \arrow[l, "j \otimes 1"'] \arrow[dl, "\sim"] \\
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\end{tikzcd}$$

braided, symmetric

Monoidal Fibrations

Let $\mathcal{X}$ be a category. A **monoidal fibration of fixed base $\boldsymbol{\mathcal{X}}$** is a fibration $P\colon \mathcal{A} \to \mathcal{X}$ for which the fibres $\mathcal{A}_x$ are monoidal, and the reindexing functors are monoidal.

Let $\mathrm{MonFib}(\mathcal{X})$ denote the 2-category of monoidal fibrations.

Monoidal Indexed Categories

Let $\mathcal{X}$ be a category. A **monoidal indexed category of fixed base $\boldsymbol{\mathcal{X}}$** is a pseudofunctor $\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{MonCat}$.

Let $\mathsf{MonICat}(\mathcal{X})$ denote the 2-category of monoidal indexed categories

Fixed base Monoidal Grothendieck Construction

Theorem (Vasilakopoulou and M [MV19])

The Grothendieck construction lifts to 2-equivalences

$$\mathsf{MonFib}(\mathcal{X}) \simeq \mathsf{MonICat}(\mathcal{X})$$

between the 2-categories of monoidal fibrations and monoidal indexed categories.

Monoidal Fibrations

A **monoidal fibration** is a fibration $P\colon \mathcal{A} \to \mathcal{X}$ for which both the total $\mathcal{A}$ and base category $\mathcal{X}$ are monoidal, P is a strict monoidal functor and the the tensor product $\otimes_{\mathcal{A}}$ of $\mathcal{A}$ preserves cartesian liftings.

Let MonFib denote the 2-category of monoidal fibrations.

Monoidal Indexed Categories

A **monoidal indexed category** is a lax monoidal pseudofunctor $(\mathcal{F}, \phi, \phi_0) \colon (\mathcal{X}^{\mathrm{op}}, \otimes^{\mathrm{op}}, I) \to (\mathsf{Cat}, \times, \mathbf{1})$, where $(\mathcal{X}, \otimes, I)$ is a monoidal category.

Let MonICat denote the 2-category of monoidal indexed categories.

Monoidal Grothendieck Construction

Theorem (Vasilakopoulou and M [MV19])

The Grothendieck construction lifts to 2-equivalences

$$\mathsf{MonFib} \simeq \mathsf{MonICat}$$

between the 2-categories of monoidal fibrations and monoidal indexed categories.

Monoidal structure on the total category

Given a lax monoidal functor

$$(\mathcal{F}, \phi, \phi_0) \colon (\mathcal{X}^{\mathrm{op}}, \otimes^{\mathrm{op}}, I) \to (\mathsf{Cat}, \times, \mathbf{1})$$

$$\phi \colon \mathcal{F}x \times \mathcal{F}y \to \mathcal{F}(x \otimes y)$$

$$(x, a) \otimes (y, b) =$$

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$$(x, a) \otimes (y, b) = (x \otimes y, \phi_{x,y}(a, b))$$

Future Work

- Closed Monoidal Grothendieck Construction
- Colored Petri Nets
- modular tensor categories
- enriched Grothendieck Construction

John Baez, John Foley, and Joseph Moeller.
Network models from Petri nets with catalysts.
arXiv:1904.03550 [math.CT], 2019.

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