

Monoidal Grothendieck Construction

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UCR Math Connections

18 May 2019

Outline

1. Classical Grothendieck Construction
2. Fibre-wise Monoidal Version
3. Global Monoidal Version
4. Applications

Motivation

$$k \in \text{Ring}$$

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$$\mathbf{Mod}_{all} ???$$

Motivation

Grothendieck: Yes!

- ▶ objects (k, M) , where $M \in \text{Mod}_k$
- ▶ maps $(f, g): (k, M) \rightarrow (k', M')$
where $f: k \rightarrow k'$ and
 $g: M \rightarrow f^*(M')$



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Given

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we defined Mod_{all} to have

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Given

$$\mathcal{F}: \mathcal{X}^{\text{op}} \rightarrow \text{Cat}$$

we define $\int \mathcal{F}$ to have

- ▶ objects (x, a) , where $x \in \mathcal{X}$, $a \in \mathcal{F}(x)$
- ▶ maps $(f, g): (x, a) \rightarrow (x', a')$ where $f: x \rightarrow x'$ and $g: a \rightarrow \mathcal{F}f(a')$

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- ▶ An **indexed functor** is a natural transformation $\alpha: \mathcal{F} \Rightarrow G$
- ▶ This defines a category $\text{ICat}(\mathcal{X})$.

Example: Rings and Modules

$$\mathrm{Mod}: \mathrm{Ring}^{\mathrm{op}} \rightarrow \mathrm{Cat}$$

$$f: k \rightarrow k' \rightsquigarrow f^*: \mathrm{Mod}_{k'} \rightarrow \mathrm{Mod}_k$$

$$f^*(M) = M$$

but with the k -action defined by

$$r.m = f(r).m \text{ for } r \in k$$

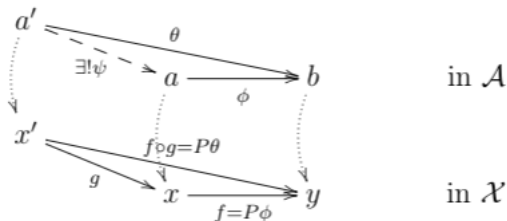
Fibred Categories

$$P: \mathcal{A} \rightarrow \mathcal{X}$$

$\phi: a \rightarrow b$ in $\mathcal{A}$ is **cartesian** if given

- ▶ $g: x' \rightarrow x$ in $\mathcal{X}$
- ▶ $\theta: a' \rightarrow b$ in $\mathcal{A}$
- ▶ with $P\theta = f \circ g$

then $\exists! \psi: a' \rightarrow a$ such that $P\psi = g$ and $\theta = \phi \circ \psi$



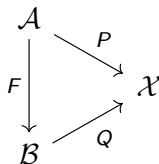
Fibred Categories

- ▶ For $x \in \text{ob } \mathcal{X}$, the **fibre of x** is $\mathcal{A}_x := P^{-1}(x)$.
- ▶ $P: \mathcal{A} \rightarrow \mathcal{X}$ is called a **fibration** if and only if, for all $f: x \rightarrow y$ in $\mathcal{X}$ and $b \in \mathcal{A}_y$, there is a **cartesian lifting** of b along f , $\phi: a \rightarrow b$.
- ▶ The category $\mathcal{X}$ is then called the **base** of the fibration, and $\mathcal{A}$ its **total category**.

Category of Fibrations

$\text{Fib}(\mathcal{X})$:

- ▶ an object is a fibration $P: \mathcal{A} \rightarrow \mathcal{X}$
- ▶ a morphism is a functor F



which preserves cartesian liftings

Example: Graphs

$$\begin{array}{ccc} \text{Grph} & & E \\ \downarrow \scriptstyle V & & \downarrow \\ \text{Set} & X \xrightarrow{f} Y & Y \times Y \end{array}$$

Example: Graphs

Grph



Set

$$\begin{array}{ccc} Q & \xrightarrow{\quad} & E \\ \downarrow & \lrcorner & \downarrow \\ X \times X & \xrightarrow{f \times f} & Y \times Y \\ & \xrightarrow{f} & Y \end{array}$$

The Grothendieck Construction

For indexed category $F: \mathcal{X}^{\text{op}} \rightarrow \text{Cat}$, $\int \mathcal{F}$ is naturally fibred over $\mathcal{X}$:

$$\begin{aligned} P_{\mathcal{F}}: \int \mathcal{F} &\rightarrow \mathcal{X} \\ (x, a) &\mapsto x \\ (f, k) &\mapsto f \end{aligned}$$

Equivalence

Theorem

This construction gives an equivalence.

$$\mathrm{ICat}(\mathcal{X}) \cong \mathrm{Fib}(\mathcal{X})$$

Example: semidirect product

$$A: G \rightarrow \text{Aut}(H)$$

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$$\int A = H \rtimes G$$

Monoidal Categories

A **monoidal category** $\mathcal{C}$ is a category equipped with a multiplication $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a unit object I which are associative and unital up to coherent isomorphism.

Monoidal Functors

A monoidal functor

$$(F, \phi): (\mathcal{C}, \otimes) \rightarrow (\mathcal{D}, \boxtimes)$$

consists of

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

$$\phi_{x,y}: F(x) \boxtimes F(y) \xrightarrow{\sim} F(x \otimes y)$$

Monoidal Fibrations

A **monoidal fibration of fixed base $\mathcal{X}$**

- ▶ fibration $P: \mathcal{A} \rightarrow \mathcal{X}$
- ▶ the fibres $\mathcal{A}_x$ are monoidal
- ▶ the reindexing functors are monoidal

Let $\text{MonFib}(\mathcal{X})$ denote the category of monoidal fibrations.

Monoidal Indexed Categories

A **monoidal indexed category of fixed base $\mathcal{X}$** is

- ▶ a functor $\mathcal{F}: \mathcal{X}^{\text{op}} \rightarrow \text{MonCat}$

Let $\text{MonICat}(\mathcal{X})$ denote the category of monoidal indexed categories

Fixed base Monoidal Grothendieck Construction

Theorem (Vasilakopoulou and M [MV19])

The Grothendieck construction lifts to equivalences

$$\mathrm{MonFib}(\mathcal{X}) \simeq \mathrm{MonICat}(\mathcal{X})$$

between the categories of monoidal fibrations and monoidal indexed categories.

Monoidal Fibrations

A **monoidal fibration** is a fibration $P: \mathcal{A} \rightarrow \mathcal{X}$

- ▶ $\mathcal{A}$ and $\mathcal{X}$ are monoidal
- ▶ P is a strict monoidal functor
- ▶ $\otimes_{\mathcal{A}}$ preserves cartesian liftings.

Let MonFib denote the category of monoidal fibrations.

Lax Monoidal Functors

A lax monoidal functor

$$(F, \phi): (\mathcal{C}, \otimes) \rightarrow (\mathcal{D}, \boxtimes)$$

consists of

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

$$\phi_{x,y}: F(x) \boxtimes F(y) \rightarrow F(x \otimes y)$$

Monoidal Indexed Categories

A **monoidal indexed category** is

- ▶ lax monoidal $(\mathcal{F}, \phi): (\mathcal{X}^{\text{op}}, \otimes) \rightarrow (\text{Cat}, \times)$
- ▶ $(\mathcal{X}, \otimes)$ is monoidal

Let MonICat denote the category of monoidal indexed categories.

Monoidal Grothendieck Construction

Theorem (Vasilakopoulou and M [MV19])

The Grothendieck construction lifts to equivalences

$$\mathbf{MonFib} \simeq \mathbf{MonICat}$$

between the categories of monoidal fibrations and monoidal indexed categories.

Monoidal structure on the total category

Given a lax monoidal functor

$$(\mathcal{F}, \phi): (\mathcal{X}^{\text{op}}, \otimes) \rightarrow (\text{Cat}, \times)$$

$$\phi: \mathcal{F}x \times \mathcal{F}y \rightarrow \mathcal{F}(x \otimes y)$$

$$(x, a) \otimes (y, b) =$$

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$$\phi: \mathcal{F}x \times \mathcal{F}y \rightarrow \mathcal{F}(x \otimes y)$$

$$(x, a) \otimes (y, b) = (x \otimes y, \phi_{x,y}(a, b))$$

Example: Modules

$$\begin{aligned}\mathrm{Mod} &: \mathrm{Ring}^{\mathrm{op}} \rightarrow \mathrm{Cat} \\ (f: k \rightarrow k') &\mapsto (f^*: \mathrm{Mod}_{k'} \rightarrow \mathrm{Mod}_k)\end{aligned}$$

Example: Modules

$$\mathbf{Mod} : \mathbf{Ring}^{\mathrm{op}} \rightarrow \mathbf{Cat}$$

$$(f : k \rightarrow k') \mapsto (f^* : \mathbf{Mod}_{k'} \rightarrow \mathbf{Mod}_k)$$

$\int \mathbf{Mod} :$

$$(k, M) \xrightarrow{(f, g)} (k', N)$$

Example: Modules

$$(\mathrm{Mod}, \mu): (\mathrm{Ring}^{\mathrm{op}}, \otimes) \rightarrow (\mathrm{Cat}, \times)$$

$$\begin{aligned} \mu: \mathrm{Mod}_k \times \mathrm{Mod}_{k'} &\rightarrow \mathrm{Mod}_{k \otimes k'} \\ (M, N) &\mapsto M \otimes_{\mathbb{Z}} N \end{aligned}$$

Example: Modules

$$(\mathbf{Mod}, \mu): (\mathbf{Ring}^{\mathrm{op}}, \otimes) \rightarrow (\mathbf{Cat}, \times)$$

$$\begin{aligned}\mu: \mathbf{Mod}_k \times \mathbf{Mod}_{k'} &\rightarrow \mathbf{Mod}_{k \otimes k'} \\ (M, N) &\mapsto M \otimes_{\mathbb{Z}} N\end{aligned}$$

$$(\int \mathbf{Mod}, \otimes_{\mu}):$$

$$(k, M) \otimes_{\mu} (k', N) = (k \otimes k', M \otimes_{\mathbb{Z}} N)$$



Joe Moeller and Christina Vasilakopoulou.

Monoidal grothendieck construction.

[arXiv:1809.00727 \[math.CT\]](https://arxiv.org/abs/1809.00727), 2019.