Monoidal Grothendieck Construction

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UCR Math Connections
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Outline

1. Classical Grothendieck Construction
2. Fibre-wise Monoidal Version
3. Global Monoidal Version
4. Applications

Motivation

$k \in \mathsf{Ring}$

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$$k \in \mathsf{Ring} \rightsquigarrow \mathsf{Mod}_k \in \mathsf{Cat}$$

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$$\mathsf{Mod}_{all}???$$

Motivation

Grothendieck: Yes!

- objects (k, M), where $M \in \mathrm{Mod}_k$
- maps $(f, g)\colon (k, M) \to (k', M')$ where $f\colon k \to k'$ and $g\colon M \to f^*(M')$

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$$k \in \mathsf{Ring} \rightsquigarrow \mathsf{Mod}_k \in \mathsf{Cat}$$

$$\mathsf{Mod} \colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$$

$$f \colon k \to k' \rightsquigarrow f^* \colon \mathsf{Mod}_{k'} \to \mathsf{Mod}_k$$

Motivation

Given

$$\mathrm{Mod}\colon \mathrm{Ring}^{\mathrm{op}} \to \mathrm{Cat}$$

we defined Mod_{all} to have

- objects (k, M), where $M \in \mathrm{Mod}_k$
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Given

$$\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

we define $\int \mathcal{F}$ to have

- objects (x, a), where $x \in \mathcal{X}$, $a \in \mathcal{F}(x)$
- maps $(f, g)\colon (x, a) \to (x', a')$ where $f\colon x \to x'$ and $g\colon a \to \mathcal{F}f(a')$

Indexed Categories

Let $\mathcal{X}$ be a category.

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- An **indexed category** is a functor $\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$.
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- This defines a category $\mathsf{ICat}(\mathcal{X})$.

Example: Rings and Modules

$$\mathsf{Mod}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$$

$$f\colon k \to k' \rightsquigarrow f^*\colon \mathsf{Mod}_{k'} \to \mathsf{Mod}_k$$

$$f^*(M) = M$$

but with the k-action defined by

$$r.m = f(r).m \text{ for } r \in k$$

Fibred Categories

$$P\colon \mathcal{A} \to \mathcal{X}$$

$\phi\colon a \to b$ in $\mathcal{A}$ is **cartesian** if given

- $g\colon x' \to x$ in $\mathcal{X}$
- $\theta\colon a' \to b$ in $\mathcal{A}$
- with $P\theta = f \circ g$

then $\exists!\, \psi\colon a' \to a$ such that $P\psi = g$ and $\theta = \phi \circ \psi$

$$\begin{tikzcd}
a' \arrow[drr, "\theta"] \arrow[dr, dashed, "\exists!\psi"'] \arrow[dd, dotted] & & & \\
& a \arrow[r, "\phi"'] \arrow[dd, dotted] & b \arrow[dd, dotted] & \text{in } \mathcal{A} \\
x' \arrow[dr, "g"'] \arrow[drr, "f\circ g = P\theta"] & & & \\
& x \arrow[r, "f = P\phi"'] & y & \text{in } \mathcal{X}
\end{tikzcd}$$

Fibred Categories

- For $x \in \mathrm{ob}\,\mathcal{X}$, the **fibre of *x*** is $\mathcal{A}_x := P^{-1}(x)$.
- $P\colon \mathcal{A} \to \mathcal{X}$ is called a **fibration** if and only if, for all $f\colon x \to y$ in $\mathcal{X}$ and $b \in \mathcal{A}_Y$, there is a **cartesian lifting** of b along f, $\phi\colon a \to b$.
- The category $\mathcal{X}$ is then called the **base** of the fibration, and $\mathcal{A}$ its **total category**.

Category of Fibrations

$\mathrm{Fib}(\mathcal{X})$:

- an object is a fibration $P\colon \mathcal{A} \to \mathcal{X}$
- a morphism is a functor F

$$\begin{tikzcd} \mathcal{A} \arrow[dr, "P"] \arrow[dd, "F"'] & \\ & \mathcal{X} \\ \mathcal{B} \arrow[ur, "Q"'] & \end{tikzcd}$$

which preserves cartesian liftings

Example: Graphs

$$\begin{tikzcd} \mathrm{Grph} \arrow[d, "V"'] \\ \mathrm{Set} \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[d] \\ & Y \times Y \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Example: Graphs

$$
\begin{tikzcd}
\mathsf{Grph} \arrow[dd, "V"'] & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\
 & X \times X \arrow[r, "f \times f"'] & Y \times Y \\
\mathsf{Set} & X \arrow[r, "f"] & Y
\end{tikzcd}
$$

The Grothendieck Construction

For indexed category $F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$, $\int \mathcal{F}$ is naturally fibred over $\mathcal{X}$:

$$\begin{aligned} P_{\mathcal{F}}\colon \int \mathcal{F} &\to \mathcal{X} \\ (x, a) &\mapsto x \\ (f, k) &\mapsto f \end{aligned}$$

Equivalence

Theorem

This construction gives an equivalence.

$$\mathsf{ICat}(\mathcal{X}) \cong \mathsf{Fib}(\mathcal{X})$$

Example: semidirect product

$$A\colon G \to \mathrm{Aut}(H)$$

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$$\int A = H \rtimes G$$

Monoidal Categories

A **monoidal category** $\mathcal{C}$ is a category equipped with a multiplication $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object I which are associative and unital up to coherent isomorphism.

Monoidal Functors

A **monoidal functor**

$$(F,\phi)\colon (\mathcal{C},\otimes) \to (\mathcal{D},\boxtimes)$$

consists of

$$F\colon \mathcal{C} \to \mathcal{D}$$

$$\phi_{x,y}\colon F(x)\boxtimes F(y) \xrightarrow{\sim} F(x\otimes y)$$

Monoidal Fibrations

A **monoidal fibration of fixed base $\mathcal{X}$**

- fibration $P\colon \mathcal{A} \to \mathcal{X}$
- the fibres $\mathcal{A}_x$ are monoidal
- the reindexing functors are monoidal

Let $\mathrm{MonFib}(\mathcal{X})$ denote the category of monoidal fibrations.

Monoidal Indexed Categories

A **monoidal indexed category of fixed base $\mathcal{X}$** is

- a functor $\mathcal{F} \colon \mathcal{X}^{\mathrm{op}} \to \mathsf{MonCat}$

Let $\mathsf{MonICat}(\mathcal{X})$ denote the category of monoidal indexed categories

Fixed base Monoidal Grothendieck Construction

Theorem (Vasilakopoulou and M [MV19])

The Grothendieck construction lifts to equivalences

$$\mathsf{MonFib}(\mathcal{X}) \simeq \mathsf{MonICat}(\mathcal{X})$$

between the categories of monoidal fibrations and monoidal indexed categories.

Monoidal Fibrations

A **monoidal fibration** is a fibration $P\colon \mathcal{A} \to \mathcal{X}$

- $\mathcal{A}$ and $\mathcal{X}$ are monoidal
- P is a strict monoidal functor
- $\otimes_{\mathcal{A}}$ preserves cartesian liftings.

Let MonFib denote the category of monoidal fibrations.

Lax Monoidal Functors

A **lax monoidal functor**

$$(F, \phi) \colon (\mathcal{C}, \otimes) \to (\mathcal{D}, \boxtimes)$$

consists of

$$F \colon \mathcal{C} \to \mathcal{D}$$

$$\phi_{x,y} \colon F(x) \boxtimes F(y) \to F(x \otimes y)$$

Monoidal Indexed Categories

A **monoidal indexed category** is

- lax monoidal $(\mathcal{F}, \phi)\colon (\mathcal{X}^{\mathrm{op}}, \otimes) \to (\mathsf{Cat}, \times)$
- $(\mathcal{X}, \otimes)$ is monoidal

Let MonICat denote the category of monoidal indexed categories.

Monoidal Grothendieck Construction

Theorem (Vasilakopoulou and M [MV19])

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between the categories of monoidal fibrations and monoidal indexed categories.

Monoidal structure on the total category

Given a lax monoidal functor

$$(\mathcal{F}, \phi) \colon (\mathcal{X}^{\mathrm{op}}, \otimes) \to (\mathsf{Cat}, \times)$$

$$\phi \colon \mathcal{F}x \times \mathcal{F}y \to \mathcal{F}(x \otimes y)$$

$$(x, a) \otimes (y, b) =$$

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$$(x, a) \otimes (y, b) = (x \otimes y, \phi_{x,y}(a, b))$$

Example: Modules

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$\int \mathsf{Mod}$:

$$(k, M) \xrightarrow{(f,g)} (k', N)$$

Example: Modules

$$(\mathsf{Mod}, \mu) \colon (\mathsf{Ring}^{\mathrm{op}}, \otimes) \to (\mathsf{Cat}, \times)$$

$$\begin{aligned} \mu \colon \mathsf{Mod}_k \times \mathsf{Mod}_{k'} &\to \mathsf{Mod}_{k \otimes k'} \\ (M, N) &\mapsto M \otimes_{\mathbb{Z}} N \end{aligned}$$

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$(\int \mathsf{Mod}, \otimes_\mu)$:

$$(k, M) \otimes_\mu (k', N) = (k \otimes k', M \otimes_{\mathbb{Z}} N)$$

Joe Moeller and Christina Vasilakopoulou.
Monoidal grothendieck construction.
arXiv:1809.00727 [math.CT], 2019.