Network Operads

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(joint w/ J. C. Baez, J. D. Foley, and B. S. Pollard)

2017 AMS Sectional Meeting at UCR

Constructing Network Operads

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- Start with your favorite lax symmetric monoidal functor $F\colon \mathbf{S} \to \mathbf{Cat}$
- apply the symmetric monoidal Grothendieck construction to get the symmetric monoidal category $\int F$ with $\otimes_F$
- Let O_F be the endomorphism operad $\mathrm{op}(\int F)$

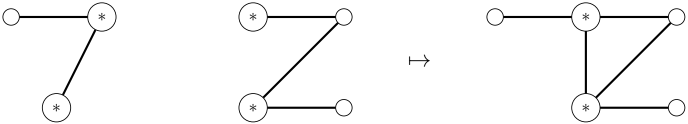
Theorem

The composite

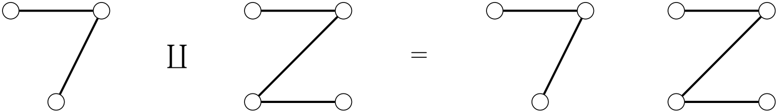
$$\mathbf{NetMod} \xrightarrow{\int} \mathbf{SSMC} \xrightarrow{\mathrm{op}(-)} \mathbf{Op}$$

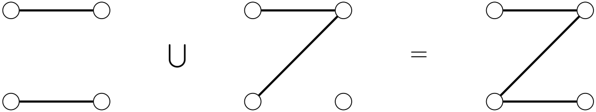
constructs a network operad for each network model.

Graphs can be combined to create bigger graphs by identifying some of the vertices

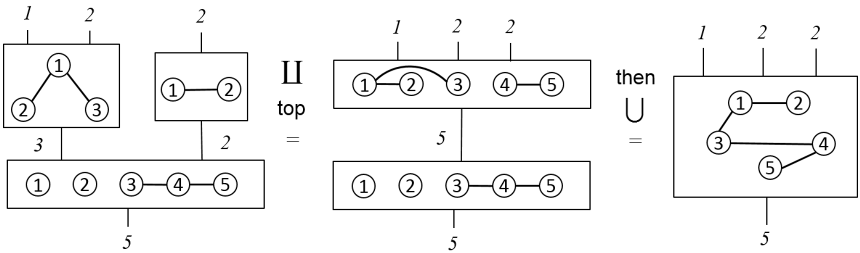


We choose to examine these as combinations of a few simpler operations





We want to construct an operad that captures these operations



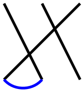
Endomorphism Operad

Given a symmetric monoidal category $(\mathbf{C}, \otimes)$, one can get a typed (aka coloured) operad $\mathrm{op}(\mathbf{C})$, called the **endomorphism operad**, with

- objects of $\mathbf{C}$ as types
- $\mathrm{op}(\mathbf{C})(c_1, \ldots, c_n; c) = \mathrm{Hom}_{\mathbf{C}}(c_1 \otimes \cdots \otimes c_n, c)$

Graph-Permutation Pairs

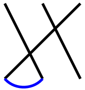
A morphism $(\sigma, g)\colon n \to n$ is a graph-permutation pair

(τ, h) 

Graph-Permutation Pairs

How should we compose two such pairs?

(τ, h)



$\circ$

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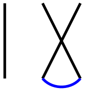

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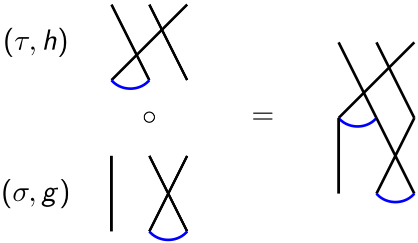

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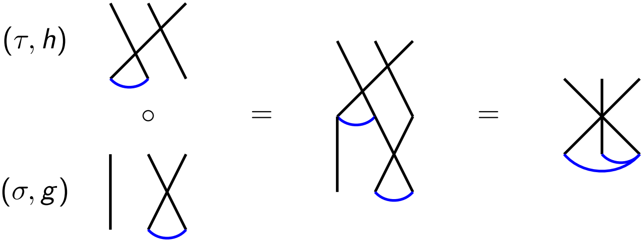

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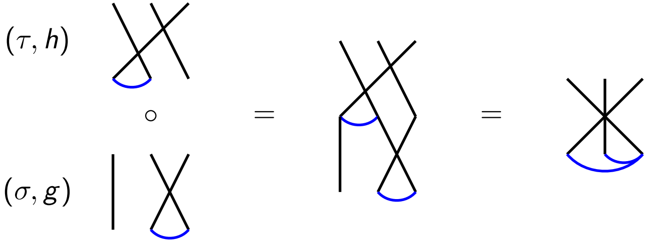
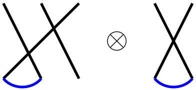


$$(\sigma, g) \circ (\tau, h) = (\sigma\tau, g \cup \sigma h)$$

Graph-Permutation Pairs

How should we tensor two such pairs?

(σ, g) (τ, h)



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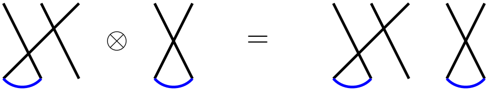

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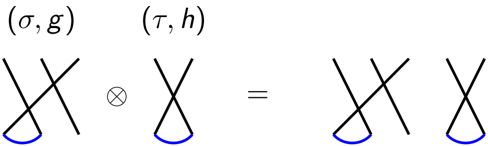
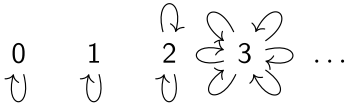


$$(\sigma, g) \otimes (\tau, h) = (\sigma + \tau, g \sqcup h)$$

The Permutation Groupoid



Definition

Let $\mathbf{S}$ *denote the category with finite sets* $\mathbf{n} = \{1, \ldots, n\}$ *as objects, and bijections for morphisms.*

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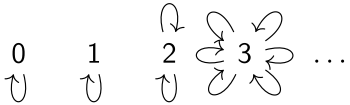

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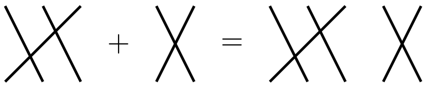
Another way to see it is $\mathbf{S} = \coprod_{n \in \mathbb{N}} S_n$.

The Permutation Groupoid

S is a symmetric monoidal category with $+$ where

- $n + m$ is exactly what you think
- $\sigma + \tau$ looks like



Simple Graphs

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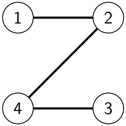
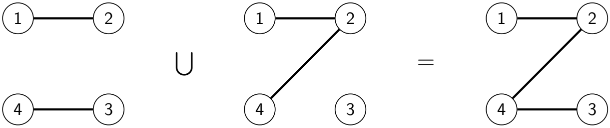
A simple graph with vertex set $\mathbf{n} = \{1, \ldots, n\}$ is a collection of subsets of $\mathbf{n}$, each of which have 2 elements. For example, this graph on $\mathbf{4}$



is the set $\{\{1,2\},\{2,4\},\{3,4\}\}$ in this setting.

Graph Monoids

Defining graphs this way allows us to take unions of graphs



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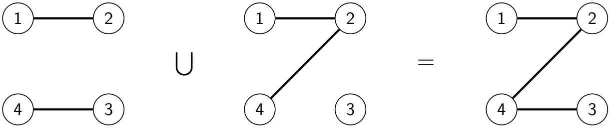

so the set of simple graphs on **n**, denoted $\mathrm{SG}(\mathbf{n})$, is a monoid with operation $\cup$.

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Putting them all together gives a functor

$$\mathrm{SG} \colon \mathbf{S} \to \mathbf{Mon}$$

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which makes SG a symmetric lax monoidal functor

$$(\mathrm{SG}, \sqcup) \colon (\mathbf{S}, +) \to (\mathbf{Cat}, \times)$$

The Grothendieck Construction

The Grothendieck construction takes a pseudofunctor

$$F \colon \mathbf{C} \to \mathbf{Cat}$$

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$$F\colon \mathbf{C} \to \mathbf{Cat}$$

and produces a category fibred over $\mathbf{C}$

$$\begin{array}{c} \int F \\ \downarrow \\ \mathbf{C} \end{array}$$

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and the total category in the fibration

$$\int F$$

The Grothendieck Construction

Given a functor $F\colon \mathbf{C} \to \mathbf{Cat}$ the Grothendieck construction gives a category $\int F$ where

- objects are pairs (c, x) where
  - c is an object in $\mathbf{C}$
  - x is an object in Fc
- morphisms are $(f, g)\colon (c, x) \to (d, y)$ where
  - $f\colon c \to d$ in $\mathbf{C}$
  - $g\colon Ff(x) \to y$ in Fd
- composition is given by

$$(f, g) \circ (f', g') = (f \circ f', g \circ Ff(g'))$$

The Monoidal Grothendieck Construction

Let $(\mathbf{C}, \otimes)$ be a monoidal category, and $F\colon \mathbf{C} \to \mathbf{Cat}$ a lax monoidal functor, with $\Phi_{c,d}\colon Fc \times Fd \to F(c \otimes d)$.

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$$(c, x) \otimes_F (d, y) = (c \otimes d, \Phi_{c,d}(x, y))$$

and

$$(f, g) \otimes_F (f', g') = (f \otimes f', \Phi_{d,d'}(g, g'))$$

The Symmetric Monoidal Grothendieck Construction

Theorem

If $F : \mathbf{C} \to \mathbf{Cat}$ *is a lax monoidal functor, there is a natural way to define a monoidal structure on* $\int F$.

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If $F \colon \mathbf{C} \to \mathbf{Cat}$ is a symmetric lax monoidal functor, there is a natural way to define a symmetric monoidal structure on $\int F$. We call this the **symmetric monoidal Grothendieck construction**.

What does this mean for simple graphs?

We said before that we want to think about simple graphs as a symmetric lax monoidal functor

$$F \colon \mathbf{S} \to \mathbf{Cat}$$

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so we can apply the symmetric monoidal Grothendieck construction to F

What does this mean for simple graphs?

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- Let O_F be the endomorphism operad $\mathrm{op}(\int F)$

Now we can generalize this

Definition

A **Network Model** *is a lax symmetric monoidal functor* $F\colon \mathbf{S} \to \mathbf{Cat}$. *Let* **NetMod** *denote the category of network models.*

Theorem

The construction

$$\mathbf{NetMod} \xrightarrow{\mathrm{op}(\int -)} \mathbf{Op}$$

is functorial.

Examples of Network Models

- Multigraphs
- Directed Graphs
- Partitions
- Graphs with colored vertices
- Petri Nets

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- Multigraphs
- Directed Graphs
- Partitions
- Graphs with colored vertices
- Petri Nets
- Graphs with edges weighted by a monoid

Further reading and acknowledgments

- Check our e-print: Network Models, arXiv:1711.00037.
- You out for postings of technical reports [2, 3] for more examples.

This work was supported by the DARPA Complex Adaptive System Composition and Design Environment (CASCADE) project.

We thank Chris Boner, Tony Falcone, Marisa Hughes, Joel Kurucar, Tom Mifflin, John Paschkewitz, Thy Tran and Didier Vergamini for many helpful discussions.

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